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DICHOTOMIES FOR BAND MATRICES.(U)

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MRC Technical Summary Report #2057

DICHOTOMIES FOR BAND MATRICES

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DICHOTOMIES FOR BAND MATRICES

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ABSTRACT

The bounded invertibility (as a linear map on  $\ell_\infty$ , say) of a bounded, strictly  $m$ -banded biinfinite matrix  $A$  is shown to be equivalent to a dichotomy or splitting of its kernel  $\mathcal{N}$  (as a map on  $\mathbb{R}^{\mathbb{Z}}$ ) into  $\mathcal{N}^+$  and  $\mathcal{N}^-$ , with  $\mathcal{N}^+$  containing those which decay exponentially at  $+\infty$ , and  $\mathcal{N}^-$  those which decay exponentially at  $-\infty$ , together with a certain uniformity (with respect to the sequence index) of this direct sum decomposition. The approximability of the solution of the biinfinite system  $Ax = b$  by solutions of finite sections of this system is characterized in terms of linear independence, uniform as  $I^* \rightarrow -(\infty)$ , of  $\mathcal{N}$  over  $I^+ \cup I^-$ , with  $I^*$  an integer interval of length  $\dim \mathcal{N}^*$ ,  $*$  = +, -.

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# SIGNIFICANCE AND EXPLANATION

Spline approximation is often most effective when the breakpoint (knot) sequence can be chosen suitably nonuniform. At the same time, standard spline approximation schemes (such as least-squares approximation by splines) are so far only known to be bounded as long as the breakpoint sequence is almost uniform. Any such bound is obtained (explicitly or implicitly) in terms of a bound on the inverse of certain matrices which are banded. Any attempt at establishing bounds for more general breakpoint sequences must therefore come to grips with the inverses of these band matrices. The hope is that Demko's discovery of the exponential decay of band matrix inverses will lead eventually to those desired bounds.

In the present report, this exponential decay is related to the exponential decay of solutions of the homogeneous problem  $Ax = 0$ . In particular, proofs are provided for the statements made in the earlier report MRC TSR 2049 entitled "What is the main diagonal of a biinfite band matrix?"

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

## DICHOTOMIES FOR BAND MATRICES

Carl de Boor

Dedicated to Garrett Birkhoff on the occasion of his 70th birthday

1. Introduction. In retrospect, the exponential decay of the Lagrange splines for cubic spline interpolation at knots proved in Birkhoff and de Boor [1] appears as the first instance in spline theory of exponential decay of band matrix inverses. Since then, the exponential decay of band matrix inverses has been used successfully by I.J.Schoenberg and others (see, e.g., [10]) in the analysis of cardinal splines in which the band matrices in question are Toeplitz matrices, hence well known to have exponentially decaying inverses. In adapting the proof of Douglas, Dupont and Wahlbin [6] for the boundedness of least-squares approximation by splines on a quasi-uniform mesh to more general spline approximation schemes, S. Demko [5] discovered that, in a nontrivial way, all band matrix inverses decay exponentially away from the main diagonal. At that time, I had used the idea behind the Douglas, Dupont and Wahlbin argument to carry some of [1] over to odd-degree spline interpolation at knots, but was pleased to find (in [2]) how nicely the idea behind Demko's argument simplified the proofs.

As a kind of afterthought, I proved in [2] that any nontrivial solution of the biinfinite homogeneous linear system  $Ax = 0$  must grow exponentially in at least one direction in case  $A$  is a band matrix, bounded and boundedly invertible (on  $\ell_\infty$ , say). With that in mind, though, further considerations of odd-degree spline interpolation at knots led to the characterization of bounded invertibility (as a map on  $\ell_\infty$ ) of a band matrix in terms of a dichotomy or splitting of its kernel (as a map on  $\mathbb{R}^2$ ). These results were stated in [3] and are restated in greater detail and proved here.

I have to confess now that I became only recently familiar with the well established theory of ordinary and exponential dichotomies for ordinary linear differential operators in Banach space, particularly through Coppel's eminently readable book [4]. I have tried to

deduce the results presented here from those in Coppel's book by reinterpreting band matrices as difference operators and then going from the discrete to the continuous. But it seemed, in the end, more satisfactory and "sachgerecht" to argue directly in terms of band matrices. Perhaps someone else will be more successful in this translation effort. In any event, Coppel's book could be an inspiration to those studying biinfinite band matrices.

Here is an outline of the paper. In Section 2, it is proved that, for a strictly  $m$ -banded matrix  $A$ , there exists an  $\ell_\infty$ -columned matrix  $A^{(-)}$  for which  $AA^{(-)} = I = A^{(-)}A$  iff the kernel,  $\mathcal{K}_A$ , of  $A$  (as a map on  $\mathbb{R}^{\mathbb{Z}}$ ) is the direct sum of  $\mathcal{K}_A^+$  and  $\mathcal{K}_A^-$ , with  $\mathcal{K}_A^+$  consisting of those elements of  $\mathcal{K}_A$  which are bounded at  $+\infty$ . In Section 3, the bounded invertibility (on  $\ell_p$ ) of a bounded strictly  $m$ -banded matrix  $A$  is shown to be equivalent to having the elements of  $\mathcal{K}_A^+$  decay exponentially at  $+\infty$  together with a certain uniformity condition. Finally, in Section 4, the approximability of the solution of the biinfinite system  $Ax = b$  by solutions of finite sections of this system is characterized in terms of linear independence, uniform as  $I^* \rightarrow -(\infty)$ , of  $\mathcal{K}_A$  over  $I^+ \cup I^-$ , with  $I^*$  an integer interval of length  $\dim \mathcal{K}_A^+$ .

2. The index of a band matrix. The  $r$ -th diagonal or band of the biinfinite matrix  $A$  is the sequence  $(A(i-r, i))_{i=-\infty}^{\infty}$ . Here and below,  $A(i, j)$  denotes the  $(i, j)$ -th entry of  $A$ . A biinfinite matrix  $A$  is banded (or, a band matrix) if all but finitely many of its bands are zero. Such a band matrix  $A$  gives rise to a linear map on  $\mathbb{R}^{\mathbb{Z}}$ , the linear space of all real biinfinite sequences, and we will identify  $A$  with that map.

A biinfinite matrix  $A$  is called  $m$ -banded if all but  $m+1$  consecutive bands are zero. (I chose the term " $m$ -banded" in preference to " $(m+1)$ -banded" since such a matrix is equivalent to an  $m$ -th order difference operator, its kernel usually has dimension no bigger than  $m$ , etc.) Thus  $A$  is  $m$ -banded iff for some  $\ell$

$$A(i-\ell, j) \neq 0 \text{ implies } i \leq j \leq i+m.$$

Unless otherwise indicated (e.g., by context), I will always assume that  $\ell = 0$ . This is merely a normalization achieved by considering  $E^{-\ell}A$  instead of  $A$ , with  $E$  the shift,

$$(Ea)(i) := a(i+1), \text{ all } i, \text{ all } a \text{ in } \mathbb{R}^{\mathbb{Z}},$$

an invertible operator which preserves more or less all interesting structures in  $\mathbb{R}^{\mathbb{Z}}$ .

The banded matrix  $A$  is called strictly  $m$ -banded if

$$(2.1) \quad A(i, i)A(i, i+m) \neq 0, \text{ all } i,$$

i.e., the first and last nontrivial band is never zero. This nontrivial assumption insures that, for every  $i$ , every  $a$  in  $\mathbb{R}^m$  gives rise to one and only one sequence  $f$  with

$$Af = 0 \text{ and } f(i+j) = a_j, j=1, \dots, m.$$

To put it differently, with

$$\mathcal{N} := \mathcal{N}_A := \{f \in \mathbb{R}^{\mathbb{Z}} : Af = 0\}$$

denoting the kernel or nullspace of  $A$ , strict  $m$ -bandedness of  $A$  insures that

$$(2.2) \quad \text{for every } i, \text{ the map } \mathcal{N} \rightarrow \mathbb{R}^m : f \mapsto f[i] \text{ is one-one and onto.}$$

Here and below,  $f[i]$  denotes the  $m$ -vector  $(f(i+1), \dots, f(i+m))$ . For a subset  $F$  of  $\mathbb{R}^{\mathbb{Z}}$ , we use, correspondingly,

$$F[i] := \{f[i] : f \in F\}.$$

Also, it will be convenient to denote by  $E_i$  the inverse of the  $i$ -th map in (2.2), i.e.,

$$(2.3) \quad \text{for all } f \in \mathcal{N} \quad E_i(f[i]) = f.$$

$E_1$  is the "fundamental solution" for the homogeneous system

$$Af = 0$$

which produces the particular solution  $\underline{f} = E_1 \underline{a}$  corresponding to the given initial values  $\underline{f}[1] = \underline{a}$ .

It is the purpose of this paper to characterize certain aspects of strictly  $m$ -banded matrices in terms of their kernel. It is therefore important to realize that such matrices are essentially determined by their kernel.

Proposition 1. If  $\mathcal{U}$  is an  $m$ -dimensional subspace of  $\mathbb{R}^{\mathbb{Z}}$  which satisfies (2.2), then there exists, up to left multiplication by an invertible diagonal matrix, exactly one strictly  $m$ -banded matrix  $A$  for which  $\mathcal{U}_A = \mathcal{U}$ .

We also introduce two subspaces of  $\mathcal{U}$ ,

$$\mathcal{U}^* := \{ \underline{f} \in \mathcal{U} : \lim_{i \rightarrow \infty} \underline{f}(i) < \infty \},$$

with  $*$  standing for either  $+$  or  $-$ . (We continue this convenient use of  $*$  throughout the paper.)

Definition. The strictly  $m$ -banded matrix  $A$  has index  $k$ , or,  $\text{index}(A) = k$ , in case  $\mathcal{U}_A = \mathcal{U}_A^+ \oplus \mathcal{U}_A^-$  and  $\dim \mathcal{U}_A^+ = k$ .

In particular, if  $A$  has index, then

$$\begin{aligned} m^+ &:= \dim \mathcal{U}_A^+ = \text{index}(A), \\ m^- &:= \dim \mathcal{U}_A^- = m - \text{index}(A). \end{aligned}$$

The notion of index is introduced here since  $A$  has index iff  $A$  is "invertible" in a certain weak sense. In the following statement of this equivalence, we use the Dirac-sequence  $\underline{\delta}_1$  defined by

$$\underline{\delta}_1(j) = \delta_{1j}.$$

Proposition 2. The strictly  $m$ -banded matrix  $A$  has index iff for every  $i$ , there



exists exactly one  $\underline{c}_i \in \ell_\infty$  with  $\underline{Ac}_i = \underline{\delta}_i$ .

This proposition is given in [3], but we give here its proof, slightly altered, for completeness.

Proof. Since  $\mathcal{N}^- \cap \mathcal{N}^+$  is the kernel of  $A|_{\ell_\infty}$ , there is at most one solution (for any particular  $i$ ) if and only if  $\mathcal{N}^- \cap \mathcal{N}^+ = \{0\}$ . Hence it is sufficient to prove that, given uniqueness, we have  $m^+ + m^- = m$  iff there is a solution for every  $i$ .

For this, note that  $\underline{c}_i$  satisfies  $\underline{Ac}_i = \underline{\delta}_i$  iff

$$(2.4) \quad \underline{c}_i = \begin{cases} \underline{c}_i^- := E_{i-1} \underline{c}_i[i-1] & \text{on } ]-\infty, i+m[ \\ \underline{c}_i^+ := E_i \underline{c}_i[i] & \text{on } ]i, \infty[ \end{cases}$$

and

$$(2.5) \quad \sum_{j=i}^{i+m} A(i, j) \underline{c}_i(j) = 1.$$

In words,  $\underline{c}_i$  is necessarily determined by the  $(m+1)$ -vector  $(\underline{c}_i(i), \dots, \underline{c}_i(i+m))$ : For  $j < i+m$ ,  $\underline{c}_i(j)$  coincides with the extension of  $\underline{c}_i[i-1] = (\underline{c}_i(i), \dots, \underline{c}_i(i+m-1))$  to an element of  $\mathcal{N}$ , while, for  $j > i$ ,  $\underline{c}_i(j)$  coincides with the extension of  $\underline{c}_i[i] = (\underline{c}_i(i+1), \dots, \underline{c}_i(i+m))$  to an element of  $\mathcal{N}$ . Consequently,  $\underline{c}_i \in \ell_\infty$  iff

$$(2.6) \quad \underline{c}_i^+ \in \mathcal{N}^+.$$

Now assume  $m^- + m^+ = m$ . Note that (2.4) - (2.6) constitute a linear system

$$(2.7) \quad P_i(\underline{c}_i|_{[i, i+m]}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

in the  $m+1$  unknowns  $\underline{c}_i(i), \dots, \underline{c}_i(i+m)$ , with the first  $m-m^-$  homogeneous equations ensuring that  $E_{i-1} \underline{c}_i[i-1]$  lies in  $\mathcal{N}^-$ , i.e.,  $\underline{c}_i[i-1] \in \mathcal{N}^-[i-1]$ , the next equation, the only inhomogeneous one, being just (2.5), and the last  $m-m^+$  homogeneous equations ensuring that  $E_i \underline{c}_i[i]$  lies in  $\mathcal{N}^+$ , i.e.,  $\underline{c}_i[i] \in \mathcal{N}^+[i]$ . Since  $m^- + m^+ = m$  by assumption, this means that (2.7) has as many equations as unknowns and, since we already

know that it has at most one solution, the existence of a solution follows.

Conversely, assuming the existence of a solution for every  $i$ , consider the maps

$$\phi^* : \mathbb{R}^{m+1} \rightarrow \mathcal{N}^* : \underline{a} \mapsto \sum_{j=0}^{\bar{m}} a_j \underline{c}_{i+j}^*$$

with  $*$  standing for  $+$  or  $-$  (as before), and  $\bar{m} := m^- + m^+$ . Then

$$\text{null } \phi^* = \bar{m}+1 - \text{rank } \phi^* > \bar{m}+1 - m^* = \begin{cases} m^-+1, & \text{if } * = + \\ m^++1, & \text{if } * = - \end{cases}.$$

Consequently, there exists  $\underline{a} \in \ker \phi^+ \cap \ker \phi^- \setminus \{0\}$ . For this  $\underline{a}$ ,

$$\underline{d} := \sum_{j=0}^{\bar{m}} a_j \underline{c}_{i+j} \neq \underline{0}$$

since  $(\underline{c}_i)$  is obviously linearly independent. On the other hand, since

$$\underline{c}_{i+j}(s) = \begin{cases} \underline{c}_{i+j}^-(s), & \text{for } s < i+m \\ \underline{c}_{i+j}^+(s), & \text{for } s > i+m \end{cases}, \quad j=0, \dots, \bar{m},$$

we find

$$\underline{d} = \begin{cases} \sum_{j=0}^{\bar{m}} a_j \underline{c}_{i+j}^- & \text{on } ]-\infty, i+m[ \\ \sum_{j=0}^{\bar{m}} a_j \underline{c}_{i+j}^+ & \text{on } ]i+m, \infty[ \end{cases}$$

and therefore, by choice of  $\underline{a}$ ,  $\underline{d}(s) = 0$  for  $s < i+m$  and  $s > i+m$ . This implies

$\bar{m} > m$ , and therefore, since by assumption  $\mathcal{N}^+ \cap \mathcal{N}^- = \{0\}$ , i.e.,  $m^+ + m^- < m$ , the conclusion  $m^+ + m^- = m$  follows. |||

Example 1. Let  $A$  be the tridiagonal matrix whose rows are alternately 1,0,2 and 2,0,1. Then the two sequences

$$\begin{aligned} \underline{f}^- &:= \dots, 0, 2^i, 0, 2^{i+1}, 0, 2^{i+2}, \dots \\ \underline{f}^+ &:= \dots, 2^{-i}, 0, 2^{-i-1}, 0, 2^{-i-2}, 0, \dots \end{aligned}$$

are both in  $\mathcal{N} = \mathcal{N}_A$  and linearly independent and, obviously,  $\mathcal{N}^* = \text{span}\{\underline{f}^*\}$ . Hence  $A$  has index 1. The construction of  $\underline{c}_i$  as in the proof boils down to wanting

$$\underline{c}_i = \begin{cases} \alpha^- \underline{f}^- & \text{on } ]-\infty, i+1] \\ \alpha^+ \underline{f}^+ & \text{on } [i+1, \infty[ \end{cases}$$

and

$$(2.8) \quad A(i, i) \underline{c}_i(i) + A(i, i+1) \underline{c}_i(i+1) + A(i, i+2) \underline{c}_i(i+2) = 1.$$

This forces  $\alpha^-$  and  $\alpha^+$  to satisfy

$$\begin{aligned} (\alpha^+ \underline{f}^+ - \alpha^- \underline{f}^-)(i+1) &= 0 \\ A(i, i+2)(\alpha^+ \underline{f}^+ - \alpha^- \underline{f}^-)(i+2) &= 1 \end{aligned}$$

the latter from (2.8) using the fact that  $\underline{f}^- \in \mathcal{N}$ . In particular, since  $\underline{f}^+$  and  $\underline{f}^-$  vanish alternately (though never together), this shows that  $\underline{c}_i^+ = 0$  iff  $\underline{c}_i^- \neq 0$ .

The proof of Proposition 2 yields the following additional facts.

(i) The nontrivial sequence  $\underline{d} = \sum a_j \underline{c}_{i+j}$  in the second part of the proof has its support in the interval  $[i+m, i+m]$ , i.e., at the single point  $i+m$ . Since  $(\underline{c}_j)$  is linearly independent, it follows that, up to scalar multiple,  $(a_j)$  is the unique (finite) sequence for which  $\sum a_j \underline{c}_{i+j}$  has its support at the single point  $i+m$ . In particular, there exists exactly one  $\underline{a}_i \in \mathbb{R}^{m+1}$  for which

$$\sum_{j=0}^m \underline{a}_i(j) \underline{c}_{i+j} = \underline{\delta}_{i+m}.$$

This says that the banded matrix  $A'$  given by

$$A'(i, j) := \underline{a}_{j-m}(i-j+m)$$

satisfies

$$\sum_k \underline{c}_k(i) A'(k, j) = \sum_k \underline{a}_{j-m}(k-j+m) \underline{c}_k(i) = \sum_k \underline{a}_{j-m}(k) \underline{c}_{k+j-m}(i) = \underline{\delta}_j(i).$$

In other words, the matrix  $A^{(-)}$  whose  $k$ -th column consists of the  $\ell_\infty$ -sequence  $\underline{c}_k$  constructed in the proposition, all  $k$ , satisfies

$$AA^{(-)} = 1 = A^{(-)}A'$$

and this implies at once that  $A' = (AA^{(-)})A' = A(A^{(-)}A') = A$ . (There is a difficulty here with the change in the order of summation, since all sums are effectively finite.) This proves

Corollary 1. If  $A$  has index, then  $A$  is invertible in the sense that there exists a matrix  $A^{(-)}$  (necessarily unique) whose columns are bounded and for which

$$AA^{(-)} = 1 = A^{(-)}A.$$

In particular,  $A^T(A^{(-)})^T = 1$ , but there is no guarantee that the columns of  $A^{(-)T}$  are bounded. For, with  $D$  any invertible diagonal matrix,  $DA$  still has (the same) index, but now

$$(DA)^{(-)} = A^{(-)}D^{-1},$$

some of whose rows may be made unbounded by proper choice of  $D$ . For example, in Example 1,  $A = A^T$  and  $A^T(-) = A^{(-)T}$  as one easily verifies, hence both  $A$  and  $A^T$  have index 1. But, with  $D = \text{diag}[\dots, 2^{i/2}, \dots]$ ,  $(DA)^T$  has no index since  $\mathcal{U}_{(DA)^T}$  contains the bounded sequence  $D^{-1}\underline{e}$ , while, e.g., with  $D = \text{diag}[\dots, 2^i, \dots]$ ,  $(DA)^T$  again has index, but now index 2. In particular, now

$$(DA)^{(-)T} \neq (DA)^T(-)$$

even though both matrices satisfy the equations

$$(DA)^TX = 1 = X(DA)^T$$

(for  $X$ ). The uniqueness assertion for  $A^{(-)}$  only covers  $\ell_\infty$ -columned matrices  $A^{(-)}$ .

$A^{(-)}$  does function as the inverse for  $A$ , at least as the inverse for  $A$  restricted to sequences of finite support.

(ii) A repeat of the second part of the proof of Proposition 2, but with  $\bar{m} = m-1$  rather than equal to  $m^+ + m^-$ , shows that (in case  $A$  has index) any sequence  $\sum_{i=1}^{\infty} a_i e_i$  with  $\underline{a} \in \ker \phi^+ \cap \ker \phi^-$  must vanish identically, hence  $\ker \phi^+ \cap \ker \phi^- = \{0\}$ . Since now  $\text{rank} \phi^* = m - \text{null} \phi^* > m - m^*$ , this implies  $\text{rank} \phi^* = m^*$  and so proves

Corollary 2. If  $A$  has index, then, for every  $i$ , the map

$$\phi^* : R^m \rightarrow \mathcal{U} : \underline{a} \mapsto \sum_{j=1}^m a_j \underline{c}_{i+j}^*$$

carries  $R^m$  onto  $\mathcal{U}^*$ .

This corollary shows that any  $m$  consecutive columns of  $A^{(-)}$  supply all the information needed to construct a basis for  $\mathcal{U}^+$  and  $\mathcal{U}^-$ . Example 1 above shows that it is usually not sufficient for this purpose to consider fewer than  $m$  consecutive columns.

We conclude from (2.4) and Corollary 2 that, for any  $i$  and any  $r > 0$ , the columns of

$$A^{(-)} \begin{pmatrix} i+1, & \dots, & i+m \\ i-r-m+1, & \dots, & i-r \end{pmatrix} \quad \text{and} \quad A^{(-)} \begin{pmatrix} i+1, & \dots, & i+m \\ i+r+1, & \dots, & i+r+m \end{pmatrix}$$

span  $\mathcal{U}^+[i]$  and  $\mathcal{U}^-[i]$ , respectively, hence have rank  $m^+$  and  $m^-$ , respectively. This implies

Corollary 3. If both  $A$  and  $A^T$  have index and  $A^{(-)T} = A^T(-)$ , then

$$\text{index}(A) + \text{index}(A^T) = m.$$

(iii) Finally, we use Corollary 2 to point out a particularly useful choice for the matrix  $B_i$  in (2.7). We know that

$$\underline{c}_{i+j}(s) = \begin{cases} \underline{c}_{i+j}^-(s) & \text{for } s < i+j+m \\ \underline{c}_{i+j}^+(s) & \text{for } s > i+j \end{cases}$$

Hence, from Corollary 2, the sequence  $\underline{c}_{i-m}[i-1], \dots, \underline{c}_{i-1}[i-1]$  must contain a basis for  $\mathcal{U}^+[i-1]$ , say  $\underline{c}_{i_1}[i-1], \dots, \underline{c}_{i_m}[i-1]$ . Since  $\mathcal{U}[i-1] = \mathcal{U}^+[i-1] \oplus \mathcal{U}^-[i-1]$ , we can therefore find  $\lambda_1^+, \dots, \lambda_{m^+}^+$  with support in  $[i, i+m-1]$  so that

$$\lambda_r^+ \underline{c}_{i_s} = \delta_{rs}, \quad r, s = 1, \dots, m^+$$

while in addition

$$\lambda_r^+ \{ \mathcal{U}^- \} = \{0\}.$$

This guarantees that

$$\lambda_r^+ \underline{c}_s = 0 \text{ for all } r \text{ and all } s > i$$

since, for  $s > i$ ,  $\underline{c}_s[i-1] = \underline{c}_s^-[i-1]$ .

Finally, the linear functional

$$\alpha_i: f \mapsto \sum_{j=i}^{i+m} a_{ij} f(j)$$

has support in  $[i, i+m]$  and carries  $\underline{c}_j$  to zero for any  $j \neq i$  (since then  $\underline{c}_j|_{[i, i+m]} = \underline{c}_j^*|_{[i, i+m]}$ ), while  $\alpha_i \underline{c}_i = 1$ . It follows that the two sequences

$\underline{c}_{i_1}, \dots, \underline{c}_{i_m}, \underline{c}_i, \underline{c}_{j_1}, \dots, \underline{c}_{j_m}$  and  $\lambda_1^+, \dots, \lambda_m^+, \alpha_i, \lambda_1^-, \dots, \lambda_m^-$  are biorthonormal. This gives

Corollary 4. If  $A$  has index, then the matrix  $B_i$  in (2.7) can be chosen so that  $B_i^{-1}$  is a submatrix of  $A^{(-)}$ , specifically, so that

$$B_i^{-1} = A^{(-)} \left( \begin{matrix} i, & \dots, & i+m \\ i_1, \dots, i_m, & i, & j_1, \dots, j_m \end{matrix} \right)$$

for some  $i-m \leq i_1 < \dots < i_m < i < j_1 < \dots < j_m \leq i+m$ .

Example 1 shows that we cannot count on choosing  $B_i^{-1}$  from consecutive columns of  $A^{(-)}$ .

Corollary 4 shows that, if  $A$  is boundedly invertible on  $\ell_p$ , i.e., if  $A^{(-)}$  is a bounded map on  $\ell_p$ , then  $B_i$  can be chosen bounded below uniformly in  $i$ . This will be important when it comes to characterizing bounded invertibility of an  $m$ -banded matrix.

As a further illustration, let now  $A$  be a banded Toeplitz matrix. Then  $A$  is strictly  $m$ -banded for some  $m$ , and, without loss of generality,

$$A(i, j) = a_{j-i}, \text{ all } i, j$$

for some sequence  $(a_r)$  with  $a_r = 0$  for  $r < 0$  and  $r > m$ , and  $a_0 a_m \neq 0$ . Let  $z_1, \dots, z_m$  be the  $m$  zeros, counting multiplicity, of the characteristic polynomial  $p_A = p_a$  for  $A$ , given by

$$p_a(z) = \sum_{j=0}^m a_j z^j.$$

If these zeros are all simple, then  $\mathcal{H}_A$  is spanned by the  $m$  sequences

$$z_i := (z_i^j)_{j=-\infty}^{\infty}, \quad i=1, \dots, m.$$

In case of coincidences, sequences of the form

$$(j^r z_i^j)_{j=-\infty}^{\infty}, \quad r=0, \dots, m_i-1$$

appear, with  $m_i$  the multiplicity of  $z_i$  as a zero of  $p_a$ . Consequently,  $A$  has index iff  $|z_i| \neq 1$ , all  $i$ , i.e., iff  $p_a(z)$  does not vanish on  $|z| = 1$ , and in that case  $\text{index}(A) = \#\{z_i : |z_i| < 1\}$ . This shows that our notion of index is an extension of the index of a Toeplitz matrix as used, e.g., in Gohberg and Feldmann [7]. Further, if our Toeplitz matrix  $A$  has index, then the elements of  $\mathcal{H}_A^*$  are not only bounded at  $+\infty$ , but they actually go to zero there exponentially fast. Such exponential decay also occurs for general  $m$ -banded matrices, but only if the assumption that  $A$  have index can be strengthened to  $A$  having a bounded inverse (as a map on  $\ell_\infty$ , say).

3. Exponential decay. Exponential decay has been recognized as a characteristic feature of the bounded inverse of a linear differential operator for some time. Massera and Schäffer [8] credit Perron [9] with having first observed this (for nonautonomous systems). Further, they have introduced the term "(exponential) dichotomy" to describe the concomitant property of the kernel, i.e., of the linear space of solutions to the homogeneous problem, to break up into two subspaces, with one consisting of those solutions which are bounded (decay exponentially) at  $-\infty$ , and the other of those solutions which are bounded (decay exponentially) at  $+\infty$ . In addition, their notion of "(exponential) dichotomy" includes a certain uniformity with respect to the independent variable of this direct sum decomposition.

For the case of band matrices, this exponential decay is of course well recognized in the special case of Toeplitz matrices, i.e., of constant coefficient difference equations. But if there is a worked-out theory for general band matrices to parallel the development in, say, Coppel [4] for ordinary linear differential operators, I have not been able to find it. I have only seen discussions in which the bands are assumed to become constant

asymptotically.

All results concerning the exponential decay of inverses of band matrices can be based on the following theorem due to Demko [5], or its obvious generalizations.

Theorem 1. [Demko] If  $A$  is a finite invertible band matrix, then there exist  $\text{const}$  and  $\lambda \in ]0,1[$  which may depend on  $\|A\|$ ,  $\|A^{-1}\|$  and the band width of  $A$  but not on the order of  $A$  so that

$$(3.1) \quad |A^{-1}(i,j)| \leq \text{const } \lambda^{|i-j|}, \text{ all } i,j.$$

A simple proof which makes the constants  $\text{const}$  and  $\lambda$  explicit and makes obvious how this result would apply in a multivariate situation (i.e., when  $i, j$  range over a multidimensional grid) can be found in [2] (see the proof for Lemma 2 below). The essential feature of the argument (and the wherefore of the exponential decay) is already evident from the argument for the following lemma and its corollary (also from [2]).

Lemma 1. If  $A$  is  $m$ -banded (with  $\ell = 0$ ), and bounded and boundedly invertible as a map on  $\ell_p$  for some  $p < \infty$ , then, for any  $\underline{f} \in \ell_p$  and any  $i < j$ ,

$$(3.2) \quad \|\underline{f}\|_{[i,j]}^p \leq \frac{\kappa^p - 1}{\kappa^p + 1} \|\underline{f}\|_{[i-m,j+m]}^p$$

with

$$\kappa := \|A\| \|A^{-1}\|.$$

Proof. Let  $\underline{f}' := \underline{f}|_{[i,j]}$ ,  $\underline{f}'' := \underline{f}|_{[i-m,j+m]}$ . Then  $\text{supp } A\underline{f}' \subseteq [i-m,j]$  and  $\text{supp } A\underline{f}'' = \text{supp } A(\underline{f}-\underline{f}') \subseteq \mathbb{Z} \setminus [i-m,j]$ , hence

$$\text{supp } A\underline{f}' \cap \text{supp } A\underline{f}'' = \emptyset.$$

Therefore



$$\begin{aligned}
\|A\|^P (\|\underline{f}''\|^P - \|\underline{f}'\|^P) &= \|A\|^P \|\underline{f}'' - \underline{f}'\|^P \\
&> \|A\underline{f}'' - A\underline{f}'\|^P = \|A\underline{f}''\|^P + \|A\underline{f}'\|^P \\
&> \|A^{-1}\|^{-P} (\|\underline{f}''\|^P + \|\underline{f}'\|^P)
\end{aligned}$$

which proves (3.2). |||

Corollary 1. If  $A$  is  $m$ -banded, and bounded and boundedly invertible as a linear map on  $\ell_p$  for some  $p < \infty$ , then, for any  $\underline{f} \in \mathcal{B}_A$  and any  $i$ ,

$$(3.3) \quad \|\underline{f}[i+jm]\|_p^p \geq \text{const } \Lambda^{|j|} \|\underline{f}[i]\|_p^p$$

either for  $j = 1, 2, 3, \dots$  or else for  $j = -1, -2, -3, \dots$ , with

$$(3.4) \quad \Lambda := (\kappa^p + 1)/(\kappa^p - 1), \quad \text{const} := \frac{1}{2} (\Lambda - 1)/\Lambda.$$

Proof. Let  $a_v := \|\underline{f}[i+vm]\|_p^p$ . Then, from (3.2),

$$\Lambda \sum_{-j < v < j} a_v \leq \sum_{-j < v < j} a_v$$

hence

$$(3.5) \quad \Lambda^j a_0 \leq \sum_{-j < v < j} a_v.$$

Suppose now that (3.3) is violated for some  $j = r$  and  $j = s$  with  $r < 0 < s$ , and

assume without loss that  $r \leq -s$  and that  $j = r$  is the largest integer  $\leq -s$  for which

(3.3) is violated. Then, from (3.2) and the choice of  $r$  and  $s$ ,

$$\begin{aligned}
(3.6) \quad \sum_{r < v < s} a_v &\leq (\Lambda - 1)^{-1} (a_r + a_s) < (\Lambda - 1)^{-1} \text{const } (\Lambda^{-r} + \Lambda^s) a_0 \\
&= \frac{1}{2} (\Lambda^{-r-1} + \Lambda^{s-1}) a_0.
\end{aligned}$$

On the other hand, by the choice of  $r$  and (3.5),

$$\begin{aligned}
\sum_{r < v < s} a_v &= \sum_{r < v < -s} a_v + \sum_{-s < v < s} a_v > \text{const } \frac{\Lambda^{-r} - 1 - (\Lambda^{s-1})}{\Lambda - 1} a_0 + \Lambda^{s-1} a_0 \\
&= \frac{1}{2} (\Lambda^{-r-1} - \Lambda^{s-1}) a_0 + \Lambda^{s-1} a_0 \\
&= \frac{1}{2} (\Lambda^{-r-1} + \Lambda^{s-1}) a_0
\end{aligned}$$

which contradicts (3.6). |||

Corollary 2. If  $A$  is  $m$ -banded, and bounded and boundedly invertible on some  $\ell_p$  with  $p < \infty$ , then

$$(3.7) \quad \begin{aligned} &\text{there exist } \text{const} > 0, \Lambda > 1 \text{ so that, for all } i, \text{ all } \underline{f} \in \mathcal{U}_A^*, \\ &\|\underline{f}[i-(*)m]\| > \text{const } \Lambda^j \|\underline{f}[i]\|, \quad j=1,2,3,\dots \end{aligned}$$

The proof of Lemma 1 is based on the following observations: (i) the support of  $A\underline{g}$  is only slightly larger than that of  $\underline{g}$ , i.e.,

$$\text{supp}(A\underline{g}) \subseteq \text{supp } \underline{g} + [-m, 0];$$

and (ii) if  $\underline{f} \in \mathcal{U}_A$ , then  $\text{supp}(A\underline{g}) = \text{supp } A(\underline{f}-\underline{g})$ . The second step cannot be used any more when  $\underline{f} \notin \mathcal{U}_A$ , but the same idea still works when  $A\underline{f}$  has small support, e.g., when

$$A\underline{f} = \delta_k.$$

In this case, setting  $\underline{f}' := \underline{f}|_{\mathbb{Z} \setminus [i-m, j+m]}$ ,  $\underline{f}'' := \underline{f}|_{\mathbb{Z} \setminus [i, j]}$ , we find

$$\text{supp } A\underline{f}' \subseteq \mathbb{Z} \setminus [i-m, j]$$

$$\text{supp } A\underline{f}'' \subseteq \text{supp } A(\underline{f}-\underline{f}'') \cup \text{supp } A\underline{f} \subseteq [i-m, j] \cup \{k\}$$

so that, for  $k \in [i-m, j]$ ,  $\text{supp } A\underline{f}' \cap \text{supp } A\underline{f}'' = \emptyset$ , and therefore, concluding as in the proof of Lemma 1, we obtain

Lemma 2. If  $A$  is  $m$ -banded, and bounded and boundedly invertible as a linear map on  $\ell_p$  for some  $p < \infty$ , then, for all  $i < j$  and all  $k \in [i-m, j]$ ,  $A\underline{f} = \delta_k$  and  $\underline{f} \in \ell_p$  implies

$$(3.8) \quad \|\underline{f}|_{\mathbb{Z} \setminus [i-m, j+m]}\|^p < \frac{\kappa^{p-1}}{\kappa^{p+1}} \|\underline{f}|_{\mathbb{Z} \setminus [i, j]}\|^p, \text{ with } \kappa := \|A\| \|A^{-1}\|.$$

Corollary. If  $A$  is  $m$ -banded, bounded and boundedly invertible as a linear map on  $\ell_p$  for some  $1 < p < \infty$ , then  $(A|_{\ell_p})^{-1} = B|_{\ell_p}$  with

$$(3.9) \quad |B(i, j)| \leq \text{const } \lambda^{|i-j|}$$

for some  $\text{const}$  and some  $\lambda \in ]0, 1[$  which depend only on  $\|A\|$ ,  $\|A^{-1}\|$  and  $m$ .

Proof. Since  $\|B(\cdot, k)\|_\infty < \|A^{-1}\|_p$ , the conclusion is immediate in case  $p < \infty$ . If  $p = \infty$ , then the lemma, applied to  $A^T$ , gives the conclusion for  $B^T$  rather than  $B$ . But that is clearly enough. |||

As Demko has already stressed, such a corollary shows that  $A$  is boundedly invertible on every  $\ell_p$  if it is boundedly invertible on some  $\ell_p$ . This raises the question of the best choice of  $\lambda$  in (3.9), which, from the argument, could be chosen as

$$(3.10) \quad \lambda^m = \frac{\|A\|_p^p - \|A^{-1}\|_p^{-p}}{\|A\|_p^p + \|A^{-1}\|_p^{-p}} = 1 - 2/(\|A\|_p \|A^{-1}\|_p)^p$$

for any particular  $p$ . Since  $\|C\|_p = \|C^T\|_{p'}$ , with  $1/p + 1/p' = 1$ , while (3.9) is equivalent to the same statement for  $B^T$ , we can always choose  $\lambda$  as

$$(3.11) \quad \lambda^m = 1 - 2 / \min_{1 \leq p < \infty} (\|A\|_p \|A^{-1}\|_p)^{\min\{p, p/(p-1)\}},$$

but it is not clear to me for what  $p$  this minimum might be taken on.

We are now ready to prove the main result of this section.

Proposition 3. Let  $A$  be a bounded strictly  $m$ -banded matrix. Then  $A$  is boundedly invertible on  $\ell_p$  iff (i)  $A$  has index; (ii) for each  $i$ , the matrix  $B_i$  in (2.7) can be so chosen that  $\sup_i \|B_i^{-1}\| < \infty$ ; and (iii) the elements of  $\mathcal{U}_A^*$  decay exponentially uniformly, i.e., (3.7) holds.

Proof. The sufficiency of these conditions is immediate: (i) allows the construction of  $A^{(-)}$ , (ii) implies that, for each  $i$ ,

$$|A^{(-)}(i+j, i)| < \text{const}, \quad j=0, \dots, m,$$

with const independent of  $i$ , and (iii) then implies that

$$\|A^{(-)}(\cdot, i)[i+mj]\| \leq \text{const } \lambda^{|j|}, \quad \text{all } j$$

for some const and some  $\lambda \in [0, 1[$  independent of  $i$  or  $j$ . This shows  $A^{(-)}$  to have uniform exponential decay away from the "main" diagonal, hence  $A^{(-)}$  maps  $\ell_p$  into itself for any particular  $p \in [1, \infty]$ . Since we already know that  $AA^{(-)} = 1 = A^{(-)}A$ , this shows that

$$A^{(-)}|_{\ell_p} = (A|_{\ell_p})^{-1}, \quad 1 \leq p < \infty.$$

The necessity is a bit harder to prove, but earlier results contain all of the work. If  $(A|_{\ell_p})^{-1}$  exists, then, from the corollary to Lemma 2,  $A$  is boundedly invertible on  $\ell_\infty$ , hence  $A$  has index, and, since then  $(A|_{\ell_\infty})^{-1} = A^{(-)}|_{\ell_\infty}$ , condition (ii) follows from Corollary 4 to Proposition 2, while (iii) follows directly from Corollary 2 to Lemma 1. |||

Remark. This necessary and sufficient condition for the bounded invertibility of a strictly  $m$ -banded matrix is not quite the "exponential dichotomy" introduced by Massera and Schäffer to characterize the bounded invertibility of an ordinary linear differential operator. As Coppel [4; Chap. 2] describes it, such "exponential dichotomy" requires, in addition to (3.7), that

(3.12) for some const and all  $i$ , all  $f \in \mathcal{U}_A$ ,  $\|(Qf)[i]\| \leq \text{const } \|f[i]\|$ ,  
with  $Q$  the linear projector on  $\mathcal{U}_A$  associated with the decomposition  $\mathcal{U}_A = \mathcal{U}_A^+ \oplus \mathcal{U}_A^-$ ,  
i.e.,

$$\text{ran } Q = \mathcal{U}_A^+, \quad \text{ker } Q = \mathcal{U}_A^-.$$

In Proposition 3, this condition is replaced by condition (ii) which also prevents the angle between  $\mathcal{U}^+[i]$  and  $\mathcal{U}^-[i]$  from going to zero, but in a different way. I do not know whether (3.12) is necessary for the bounded invertibility of  $A$ , but I doubt it.

The argument does show that (i) and (iii) alone imply already the bounded invertibility of  $DA$  for some suitably chosen invertible diagonal matrix  $D$ .

4. The main diagonal. In an attempt to understand how the inverse of a biinfinite band matrix might be approximated by inverses of finite sections of  $A$ , one is naturally

led to the question of what might be the main diagonal of  $A$ . For a precise definition, we need some notation.

Let  $I, J$  be integer intervals. Then

$$A_{I,J} := A|_{I \times J} = (A(i,j))_{i \in I, j \in J}$$

denotes the corresponding section of the biinfinite matrix  $A$ . We can think of  $A_{I,J}$  simply as a  $|I| \times |J|$ -matrix. But,  $A_{I,J}$  also describes the nontrivial part of the linear map  $P_I A P_J$  with

$$(P_I \underline{a})(i) := \begin{cases} \underline{a}(i) & , i \in I \\ 0 & , i \notin I \end{cases}.$$

More precisely,  $A_{I,J}$  is the matrix representation (with respect to the canonical basis) of the linear map

$$P_I (A|_{\text{ran } P_J})$$

and we will not distinguish between these two.

Suppose that  $A$  is  $m$ -banded, bounded and boundedly invertible. Then we know that the linear system

$$(4.1) \quad \underline{Ax} = \underline{b}$$

has exactly one solution in  $\ell_p$  in case  $\underline{b} \in \ell_p$ . We may then try to approximate the solution  $\underline{x}$  by truncation, i.e., by the solution  $\underline{x}_I$ , if any, of the finite (square) linear system

$$(4.1)' \quad A_I \underline{x}_I = \underline{b}|_I, \quad \underline{x}_I = P_J \underline{x}_I$$

with

$$A_I := A_{I,J}, \quad J := I + r_I := \{i + r_I : i \in I\}$$

and  $r_I$  some integer.

Call this projection scheme suitable if (4.1)' has exactly one solution for all large  $I$  and  $\|\underline{x} - \underline{x}_I\|_p \xrightarrow{I \rightarrow \mathbb{Z}} 0$ .

Take specifically  $p < \infty$ , so that  $P_I$  converges pointwise to the identity on  $\ell_p$  as  $I \rightarrow \mathbb{Z}$ . Then (see, e.g., Gohberg and Feldmann [7; Theorem II.2.1]) the projection scheme is suitable iff

$$(4.2) \quad \lim_{I \rightarrow \infty} \|A_I^{-1}\| < \infty$$

i.e.,  $A_I^{-1}$  exists for all sufficiently large  $I$  and can be bounded independently of  $I$ .

Note that Demko's result makes it unnecessary here to specify in which p-norm we measure

$\|A_I^{-1}\|$ , since  $A$  is  $m$ -banded. The  $m$ -bandedness of  $A$  also restricts the possible choices for  $r_I$ , since  $A_I$  is trivially noninvertible unless  $0 < r_I < m$ . Actually, as we now show, for our projection scheme to be suitable we usually must have

$$r_I = m^+ \text{ for all large } I.$$

Lemma 3. Let  $A$  be  $m$ -banded, bounded and boundedly invertible. If (4.2) holds, then

there exists  $\text{const} > 0$  so that, for all large  $I =: [t^+, t^-]$ ,

$$(4.3) \quad \begin{aligned} &\text{for all } \underline{f} \in \mathcal{U}_A^+, \quad \|\underline{f}\|_{I^+} > \text{const} \|\underline{f}[t^+]\| \\ &\text{with } I^+ := I \setminus J \text{ and } I^- := (I+m) \setminus J. \end{aligned}$$

Proof. Let  $\underline{f} \in \mathcal{U}_A$ . Then, for  $i \in I$ ,

$$(A_I P_J \underline{f})(i) = (A_I \underline{f})(i) = \sum_{j \in J} A(i, j) \underline{f}(j) = - \sum_{j \in J} A(i, j) \underline{f}(j) = - \sum_{j \in I^+ \cup I^-} A(i, j) \underline{f}(j).$$

Consequently

$$\|A_I P_J \underline{f}\|_\infty \leq \|A\|_\infty \|\underline{f}\|_{I^+ \cup I^-}$$

or

$$\|\underline{f}\|_{I^+} \leq \|A_I^{-1}\|_\infty \|A\|_\infty \max\{\|\underline{f}\|_{I^+}, \|\underline{f}\|_{I^-}\}$$

in case  $A_I$  is invertible.

If now, in addition,  $\underline{f} \in \mathcal{U}_A^+$ , then Corollary 2 to Lemma 1 supplies  $\text{const} > 0$  and

$\Lambda > 1$  independent of  $\underline{f}$  so that

$$\|\underline{f}[t^+]\|_\infty > \text{const} \Lambda^{j/m} \|\underline{f}[t^+ + j]\|_\infty, \quad j=1, 2, 3, \dots$$

and therefore

$$\begin{aligned} \|\underline{f}[t^+]\|_\infty &\leq \|\underline{f}\|_{I^+} + \|\underline{f}\|_{I^-} \\ &\leq (1 + \|A_I^{-1}\|_\infty \|A\|_\infty) \max\{\|\underline{f}\|_{I^+}, \text{const} \Lambda^{-|I|/m} \|\underline{f}[t^+]\|_\infty\}. \end{aligned}$$

If now  $\lim_{|I| \rightarrow \infty} \|A_I^{-1}\| < \infty$ , then, for all sufficiently large  $|I|$ ,

$$(1 + \|A_I^{-1}\|_\infty \|A\|_\infty) \text{const} \Lambda^{-|I|/m} < 1 \text{ and } 1 + \|A_I^{-1}\|_\infty \|A\|_\infty < \text{const}'$$

and then (4.3) follows for  $\ast = +$  with  $\text{const} = \text{const}'$ .

The proof for  $\ast = -$  is analogous. |||

If  $A$  is strictly  $m$ -banded (hence  $f \in \mathcal{U}_A$  and  $f[i] = 0$  implies  $f = 0$ ), then a particular consequence of (4.3) is that

$$(4.4) \quad \mathcal{U}_A^\ast \text{ is lin. independent over } I^\ast.$$

This implies that

$$m^\ast = \dim \mathcal{U}_A^\ast \leq |I^\ast| = \begin{cases} r_I, & \text{if } \ast = + \\ m - r_I, & \text{if } \ast = - \end{cases}$$

and  $m^+ = r_I$  then follows.

These considerations motivate the following definition (see [3]).

Definition. The bounded and boundedly invertible biinfinite matrix  $A$  (as a map on  $\ell_\infty$ , say) has its  $r$ -th band as main diagonal provided  $\overline{\lim}_{t \rightarrow \infty} \|(A_{I, I+r})^{-1}\| < \infty$ .

As noted already, in that case  $A^{-1}$  is the strong limit (in any  $\ell_p$  with  $p < \infty$ ) of  $(A_{I, I+r})^{-1} p_I$ . Also,  $r = m^+$  (at least in case  $A$  is strictly  $m$ -banded).

Proposition 4. Let  $A$  be a strictly  $m$ -banded, bounded and boundedly invertible matrix. Then  $A$  has its  $r$ -th band as main diagonal iff (4.3) holds.

Proof. Lemma 3 establishes the necessity of this condition. As to the sufficiency, let  $B$  be a matrix with support in  $(I+r) \times I$ . Then  $B_{I+r, I} = A_{I, I+r}^{-1}$  if and only if

$$P = A^{-1} - N \quad \text{on } [I \cup (I+m)] \times I$$

with  $N(\cdot, k) \in \mathcal{U}_A$ , all  $k \in I$ . In other words,  $A_{I, I+r}$  is invertible iff we can find, for each  $k$  in  $I$ , an  $\underline{n}_k$  in  $\mathcal{U}_A$  which agrees with  $\underline{c}_k$ , the  $k$ -th column of  $A^{-1}$ , on  $I^+ := I \setminus (I+r)$  and on  $I^- := (I+m) \setminus (I+r)$ , and, in that case,  $(\underline{c}_k - \underline{n}_k)|_I$  provides the  $k$ -th column of the inverse.

We already noted that (4.3) implies (4.4) and, in particular, that  $r = m^+$ .

Consequently, we can find  $(\underline{q}_j)_{j \in I^+}$  in  $\mathbb{R}_A^+$  so that  $\underline{q}_j(i) = \delta_{ij}$ , all  $i, j \in I^+$ . (4.2) then implies that

$$\|\underline{q}_j[t^+]\| \leq \text{const}$$

and therefore, by Corollary 2 to Lemma 1,

$$\|\underline{q}_j[t^-]\| \leq \text{const } \lambda^{|I|}, \text{ all } j \in I^+$$

for some fixed  $\lambda \in [0, 1]$ . Analogously, there are  $\underline{q}_j \in \mathbb{R}_A^-$  so that  $\underline{q}_j(i) = \delta_{ij}$  for  $i, j \in I^-$  and these satisfy

$$\|\underline{q}_j[t^+]\| \leq \text{const } \lambda^{|I|}, \text{ all } j \in I^-.$$

This reduces the task of determining  $\underline{n}_k = N(\cdot, k)$  to solving the linear system

$$\underline{c}_k(i) = \sum_{j \in I^+ \cup I^-} n_{kj} \underline{q}_j(i), \quad i \in I^+ \cup I^-$$

and this system has a coefficient matrix which differs from the identity matrix by no more than  $\text{const } \lambda^{|I|}$ , hence is invertible for all sufficiently large  $|I|$ . Write

$$\underline{n}_k^* := \sum_{j \in I} n_{kj} \underline{q}_j. \text{ Then we have further by Corollary 2 to Lemma 1 that}$$

$$\|\underline{n}_k^*[t^* + (*j)m]\| \leq \text{const } \lambda^j (1 + \lambda^{|I|}) \|\underline{c}_k\|_I^*, \quad j=1, 2, \dots$$

It follows that

$$|A^{-1}(i, k) - A_I^{-1}(i, k)| \leq \text{const } \|A^{-1}\|_\infty \lambda^{\text{dist}(i, Z \setminus I)^m}$$

i.e.,  $A_I^{-1}$  converges exponentially fast to  $A^{-1}$  in its interior. In any event,

$$\lim_{I \rightarrow Z} \|A_{I, I+r}^{-1}\| \leq \text{const } \|A^{-1}\|_\infty. \quad |||$$

Remark. I now question whether the seemingly weaker condition used in (3) to define main diagonal, viz. that  $(A_{I, I+r})^{-1}$  exist for all large  $I$  and that

$$A^{-1}(i, j) = \lim_{I \rightarrow Z} (A_{I, I+r})^{-1}(i, j), \text{ all } i, j$$

is, in fact, equivalent to (4.3). Further, the linear independence of  $\mathbb{R}_A$  over  $I^+ \cup I^-$  implies the existence of  $(A_{I, I+r})^{-1}$ , but I do not know whether the converse holds.



Remark. The matrix of Example 1, though bounded and boundedly invertible, does not have a main diagonal since the element(s) of  $\chi_A^*$  vanish at every other point, so (4.3) cannot hold for all large  $I$ . But (4.3) does hold for a subsequence, hence its center band is main in this weaker sense. By contrast, any bounded and boundedly invertible Toeplitz matrix does have a main diagonal (as is well known, see, e.g., Gokhberg and Feldman [7]) since the elements of  $\chi_A^*$  are (possibly extended) exponential sums involving  $m^*$  "frequencies", hence  $\chi_A^*$  is linearly independent over any  $m^*$  consecutive points, and uniformly so.

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ABSTRACT (continued)

sequence index) of this direct sum decomposition. The approximability of the solution of the biinfinite system  $A\underline{x} = \underline{b}$  by solutions of finite sections of this system is characterized in terms of linear independence, uniform as  $I^* \rightarrow -(\infty)$ , of  $\mathcal{U}$  over  $I^+ \cup I^-$ , with  $I^*$  an integer interval of-length  $\dim \mathcal{U}^*$ ,  $* = +, -$ .